International Journal of Current Trends in Engineering & Technology
ISSN: 2395-3152
Volume: 02, Issue: 03 (May - 2016)

COFINITELY w-SUPPLEMENTED MODULES

ZAHRA KAMALI HAMIDABADI
Babol Noshirvani University of Technology
Bābol, Mazandaran, Iran
kamalih57@gmail.com

Abstract: - Let R be a ring and M be an R-module. Then the following statements are equivalent.

1. M is cofinitely w-supplemented.
2. Every cofinitely semi simple sub module of M has a supplement that is a direct summand.
3. Every cofinitely semi simple sub module of M has a weak supplement.
4. Every cofinitely semi simple sub module of M has a Rad-supplement.

Keywords: - Cofinite sub module; Cofinitely weak supplemented module; finitely weak supplemented module.

I. INTRODUCTION

Throughout this paper all rings are associative rings with identity and all modules are unital left R-modules. Let R be a ring and M be an R-module. The notation $N \subseteq M$ means that N is a sub module of M.A sub module K of an R-module M is called small in M (denoted by $K \ll M$) if $K + L = M$ for any sub module L of M implies $L = M$, see [7]. $RadM$ indicates the Jacobson radical of M. A module M is called semi-hollow if every finitely generated proper sub module is small in M, or $RadM = M$, see [5]. Let M be an R-module and A and B be any sub modules of M.B is called a supplement of A in M if B is minimal with respect to $M = A + B$. B is a supplement of A in M if $M = A + B$ and $A \cap B \ll B$, (see [5] 20.1). M is called supplemented if every sub module of M has a supplement in. M Artinian and semi simple modules are supplemented modules. For $A \subseteq M$, a sub module B of M is called a weak supplement of A in M if $A + B = M$ and $A \cap B \ll M$A sub module N of a module M is said to be cofinite if M/N is finitely generated. M is called a cofinitely supplemented module if every cofinite sub module of M has a supplement. We shall say that a module M is w-supplemented if every semi simple sub module of M has a supplement in M. M is called cofinitely w-supplemented if every cofinitely semi simple sub module of M has a supplement in M.

II. COFINITELY W-SUPPLEMENTED MODULES

Lemma 2.1 *Let $M = N + L$ where L is a sub module of M and N is a cofinitely semi simple sub module of M.Then $M = N' \oplus L$ for some sub module N' of N.*

Proof: - Let N be a cofinitely semi simple sub module of M.Then $N \cap L$ is direct summand in N.That is, $N = (N \cap L) \oplus N'$ for some sub module N' of N.Since $M = N + L$, then we have $M = ((N \cap L) \oplus N') + L = N' + L$. So $M = N' \oplus L$ because $(N \cap L) \cap N' = N' \cap L = 0$.

Lemma 2.2 *Let U is a cofinitely semi simple sub module of M contained in Rad (M). Then U is small in M.*

Proof:- Let $U \subseteq Rad\ (M)$, where U is cofinitely semi simple in M.Then $Soc\ (U) \subseteq Soc\ (Rad\ (M))$. Since U is cofinitely semi simple, $Soc\ (U) = U$.Then $U \subseteq Soc\ (Rad\ (M))$. By 1.2 and by ([7], 19.3), $U \ll M$.

Proposition 2.3 *Let R be a ring and M be an R-module. Then the following statements are equivalent.*

1. *M is cofinitely w-supplemented.*
2. *Every cofinitely semi simple sub module of M has a supplement that is a direct summand.*
3. *Every cofinitely semi simple sub module of M has a weak supplement.*
4. *Every cofinitely semi simple sub module of M has a Rad-supplement.*

Proof:-

(1)⇒ (2): Let N is a cofinitely semi simple sub module of M. By assumption, N has a supplement K in M for some sub module K of M. That is, $M = N + K$ and $N \cap K \ll K$. By Lemma 2.1, $M = N' \oplus K$ for some sub module

N' of N.

(2)⇒ (3): Let N is a cofinitely semi simple sub module of M.By (2), N has a supplement, so N has a weak supplement, since supplements are also weak supplements.

(3)⇒ (4): Let N is a cofinitely semi simple sub module of M. Since N has a weak supplement, then there exists a sub module K of M such that $N+K = M$ and $N \cap K \ll M$.By Lemma 2.1, $M = N' \oplus K$ for some sub module N' of N.By ([7], 19.3(5)), $N \cap K \ll K$. This implies $N \cap K \leq Rad\ (K)$. Thus K is Rad-supplement of N in M.

(4)⇒ (1): Let N is a cofinitely semi simple sub module of M. By assumption, N has a Rad-supplement K in M.Then $M = N + K$ and $N \cap K \leq Rad\ (K)$, also by ([7], 21.6(1) (i)) and considering inclusion map $i: K \to M$, we say $K \cap N \leq Rad\ (M)$. Then by Lemma 2.2, $K \cap N \ll M$.Since N is cofinitely semi simple, Lemma 2.1, $M = N' \oplus K$ for some sub module N' of N. So we get $K \cap N \ll K$ by ([7], 19.3(5)).

Proposition 2.4 *Any direct summand of a cofinitely w-supplemented module is cofinitely w-supplemented.*

Proof: - Let M *be* cofinitely w-supplemented module and N *be* a direct summand of M so that $M = N \oplus K$ for some sub module K of M. Let S be cofinitely a semi simple sub module of N. If $S = 0$, then N is trivially cofinitely w-supplemented. Let $S/= 0$, since $S \subseteq M$, then $M = S + T$ and $S \cap T \ll T$ for some sub module T of M. Then by the modular law, $N = S + (N \cap T)$ and consequently by Lemma 2.1, $N = S' \oplus (N \cap T)$ for some $S' \subseteq S$. That is, $N \cap T$ is a direct summand of N.If we are able to show that $S \cap (N \cap T) \ll N \cap T$, then we are done. Since $S \cap T \ll T$ by ([7], 19.3(4)) together with the inclusion map, $S \cap T \ll M$, and since $S \cap T \subseteq N$, then by ([7], 19.3(5)) $S \cap T \ll N$ and consequently $S \cap (N \cap T) \ll N \cap T$, because $(S \cap T) \cap N = S \cap T \subseteq N \cap T$. Therefore $N \cap T$ is a supplement of S in N.

Proposition 2.5 *Any finite direct sum of cofinitely w-supplemented modules is cofinitely w-supplemented.*

Proof: - It is sufficient to prove for the case $M = M_1 \oplus M_2$ where M_1 and M_2 are cofinitely w-supplemented modules, then result follows inductively. For $i = 1, 2$, let $p_i: M \to M_i$ be the projection map. Let L be a cofinitely semi simple sub module of M. Then so are the modules $p_1\ (L) = (L + M_2) \cap M_1$ and $p_2\ (L) = (L + M_1) \cap M_2$. Then $p_1\ (L)$ and $p_2\ (L)$ have supplements H_1 and H_2 in M_1 and M_2 respectively. $M_1 + M_2 + L$ have a supplement 0 in M. By ([6], Lemma 1.3), H_2 is a supplement of $M_1 + L$ in M.Also we may say that $(L + H_2) \cap M_1 \subseteq (L + M_2) \cap M_1 = p_1\ (L)$ means $(L + H_2) \cap M_1$ is also cofinitely semi simple, and then has a supplement K in M_1. Again applying ([6], Lemma 1.3), $H_2 + K$ is a supplement of L in M.Hence M is cofinitely w-supplemented.

Proposition 2.6 *M is cofinitely w-supplemented if and only if M is amply cofinitely w-supplemented.*

Proof: - (⇐) Obvious.

(⇒) Let $M = A + B$ where A is cofinitely semi simple. Since A is cofinitely semi simple, then so is $A \cap B$ and hence by Lemma 2.1, $M = Y_1 \oplus T$ for some sub module Y_1 of $A \cap B$ and some supplement T of $A \cap B$ in M. By the modular law, $A \cap B = Y_1 \oplus (A \cap B \cap T)$. Let call $A \cap B \cap T = S$ and by applying the modular law once more to $M = Y_1 \oplus T$, we get $B = Y_1 \oplus (B \cap T)$. Let call $B \cap T = Y_2$. We consider the projection mapping $\pi: Y_1 \oplus Y_2 \to Y_2$, then $A \cap B = Y_1 \oplus S = Y_1 \oplus (A \cap (B \cap T)) = Y_1 \oplus (A \cap Y_2) = Y_1 \oplus (A \cap Y_2 \cap B)$ since $Y_2 \subseteq B$. Then $A \cap Y_2 = A \cap B \cap Y_2 = \pi\ (A \cap B) = \pi\ (Y_1 + S) = \pi(S)$. Then by ([7], 19.3(4)), $A \cap Y_2 = \pi(S)$ is small in $\pi\ (T)$ and consequently in Y_2. Also $M = A + B = A + Y_1 + Y_2 = A + Y_2$. Therefore A has supplement Y_2 contained in B.

Lemma 2.7 *Let M be an R-module with Soc (M) ≪ M, then M is cofinitely w-supplemented.*

Proof. Obviously, if $Soc\ (M) = 0$, then M is cofinitely w-supplemented. Let N be a nonzero cofinitely semi simple sub module of M, then $N \subseteq Soc(M)$, so N is small in M too, then $M = M + N$ and $M \cap N = N \ll M$.

Lemma 2.8 M *is cofinitely w-supplemented if and only if Soc (M) has a supplement in M.*

Proof: - (⇒) Straightforward.
(⇐) Let V be a supplement of Soc (M) in M, then $M = Soc(M) + V$ and $Soc(V) = Soc(M) \cap V \ll V$, then by Lemma 2.7, V is cofinitely w-supplemented. Since $M = S \oplus V$ where S is a cofinitely semi simple sub module of M by Lemma 2.1, then by 2.5, M is cofinitely w-supplemented. A sub module N of a module M is said to be radical, if $Rad(N) = N$.

Proposition 2.9 *Every radical module M is cofinitely w-supplemented.*

Proof: - Let M be a radical module, that is, $M = Rad(M)$, then $Soc(M) = Soc(Rad(M)) \ll M$ by 2.2 and then by 2.7, M is cofinitely w-supplemented.

REFERENCES

[1]. R. Alizade, G. Bilhan and P. F. Smith, *Modules whose maximal sub module have supplements, Comm. Algebra.* 29, 2389−2450, (2001).

[2]. R. Alizade, E. Büyükasik, *Cofinitely Weak Supplemented Modules. Communication in Algebra.* 31, 5377 − 5390 (2003).

[3]. G. Bilhan, *Totally Cofinitely Supplemented Modules. Int. Electron. J. Algebra.* 2, 106 − 113, (2007).

[4]. E. Büyükasik, E. Mermut and S. Ozdemir, *Rad-supplemented modules, Rend. Sem. Mat. Univ. Padova, Vol.* 124, 157 − 177, (2010).

[5]. J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, *Lifting Modules. Birkhauser* 2006.

[6]. A. Harmanc, D. Keskin, P. F. Smith, *On ⊕-supplemented Modules. Acta Math. Hungarica. 83, 161.169 (1999).*

[7]. R. Wisbauer, *Foundations of Module and Ring Theory, Gordon and Breach, Philadelphia,* (1991).